Differential Dynamical Systems in electrical circuits

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Abstract ***In this work is presented an overview about mathematical methods of differential dynamical systems in circuits theory. These methods are shortly presented and there are given indications for/about the applications.***

Keywords: ***van der Pol equation, Duffing equation, nonlinear circuits***

I. INTRODUCTION

Many physical phenomena are modeled by nonlinear systems of ordinary diferential equations. An important problem in the study of nonlinear systems is to obtain exact solutions and explicitly describe traveling wave behaviors. Modern theories describe traveling waves and coherent structures in many fields, including general relativity, high energy particle physics, plasmas, atmosphere and oceans, animal dispersal, random media, chemical reactions, biology, nonlinear electrical circuits, and nonlinear optics. For example, in nonlinear optics, the mathematics developed for the propagation of information via optical solitons is quite striking, with extremely high accuracy. It has been experimentally veried, with a span of twelve orders of magnitude: from the wavelength of light to transoceanic distances. It also guides the practical applications in modern telecommunications. Many other nonlinear wave theories mentioned above have also achieved similar success.

II. THE DUFFING EQUATION

The Duffing equation is a non-linear second-order differential equation. It is an example of a dynamical system that exhibits chaotic behaviour. The equation is given by

$$\ddot{x}+\delta\dot{x}+\beta x+\alpha x^3=\gamma\cdot\cos\omega t \qquad (1)$$

or as a system of equations.

A periodic orbit corresponds to a special type of solution for a dynamical system, namely one that repeats itself in time. A dynamical system exhibiting a stable periodic orbit is often called an **oscillator**.

Figure 2 shows the periodic orbit that exists for the vector field:

$$\frac{dx}{dt}=\alpha x-y-\alpha x\left(x^2+y^2\right)$$

$$\frac{dy}{dt}=x+\alpha y-\alpha y\left(x^2+y^2\right) \qquad (2)$$

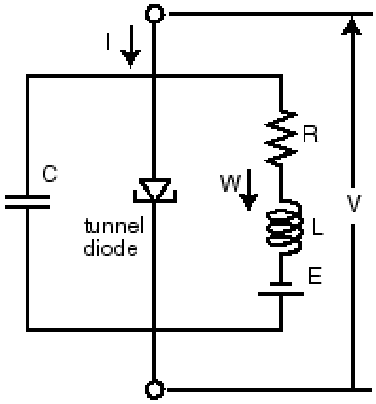

Fig. 1. Oscillator circuit

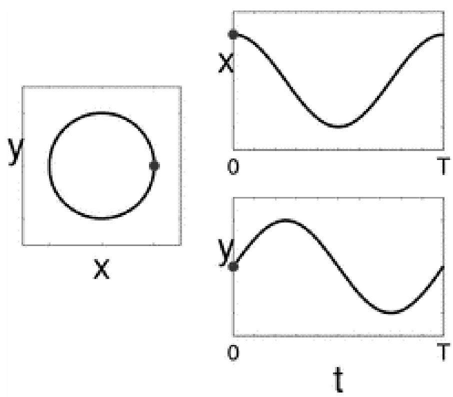

Fig.2. The periodic orbit

where α > 0 is a parameter. Transforming to radial coordinates, we see that the periodic orbit lies on a circle with unit radius for any α > 0:

$$\frac{dr}{dt}=\alpha r\left(1-r^2\right),\ \frac{d\theta}{dt}=1 \qquad (3)$$

This periodic orbit is a stable limit cycle for α > 0 and unstable limit cycle for α < 0. When α = 0, the system above has infinite number of periodic orbits and no limit cycles. For the Example above, the radial line given by θ = 0 is a Poincare section, parameterised by r.

The corresponding Poincare map $r_{i+1}=g(r_i)$ along this section may be found by explicitly integrating the vector field:

$$g(r_i)=\left[1+e^{-4\pi\alpha}\left(r_i^{-2}-1\right)\right]^{-1/2} \quad (4)$$

with fixed point $r_f=1$ corresponding the periodic orbit. Linearizing we find $g'(r_f)=e^{-4\pi\alpha}$. So, the periodic orbit is stable for any α > 0 and is unstable for any α < 0.

An alternative way to determine the stability of a periodic orbit is to use Floquet theory, which involves the time-dependent (and T-periodic) vector field linearized around the periodic orbit. Solutions to these linearized equations are used to define n Floquet multipliers characterizing the growth or decay of perturbations to the periodic orbit. It can be shown that the (n-1) eigenvalues of D_g are equal to (n-1) of the Floquet multipliers of the periodic orbit; the remaining Floquet multiplier is equal to unity and corresponds to a perturbation along the periodic orbit [Guckenheimer and Holmes,1983]. The determination of Floquet multipliers or the eigenvalues of D_g typically must be done numerically.

Given a point x_f on the periodic orbit Γ as discussed above, the eigenvalues of the matrix $D_g(x_f)$ can be used to partition the (n-1) -dimensional subspace Σ into a direct sum of subspaces $\Sigma^s \oplus \Sigma^c \oplus \Sigma^u$, corresponding to eigenvalues with modulus less than 1, equal to 1, and greater than 1, respectively. If sections Σ_x are chosen to vary continuously over different base points $x \in \Gamma$, then concatenations of the corresponding subspaces Σ_x^s, Σ_x^c, Σ_x^u form vector bundles over Γ. Stable, centre and unstable manifolds of Γ can be defined as graphs over these vector bundles.

For a non-autonomous vector field $dx/dt=f(x,t)$ with $f(x,t)=f(x,t+\tau)$ for some 0 < τ < ∞, the calculation of the stability properties of a periodic orbit with period $T=p\tau/q$, where p and q are integers can be done by considering a stroboscopic map which takes:

$$x(t)\rightarrow x(t+p\tau/q) \quad (5)$$

Stability properties follow from this map eigenvalues.

To determine the stability properties of a periodic orbit for a mapping $x_{i+1}=g(x_i)$, one can exploit the fact that a point p_0 on a period-k periodic orbit of the map g is a fixed point of the map g^k. The stability properties of this fixed point of g^k are the same as the stability properties of the periodic orbit of the map g [Guckenheimer and Holmes, 1983].

III. THE UNFORCED SYSTEM

In this section, the dynamics of the unforced system ($\gamma=0$) is examined. When there is no damping ($\delta=0$), the Duffing equation can be integrated as:

$$E(t)=\frac{1}{2}\dot{x}^2+\frac{1}{2}\beta x^2+\frac{1}{4}\alpha x^4=const. \quad (6)$$

Therefore, in this case, Duffing equation is a Hamiltonian system. The shape of E(t) for α >0 can be observed to be single-well potential for β>0 and double -well potential for β<0. Trace of $\mathbf{x}\equiv\left(x,\dot{x}\right)$ moves on the surface of E(t) keeping E(t) constant.

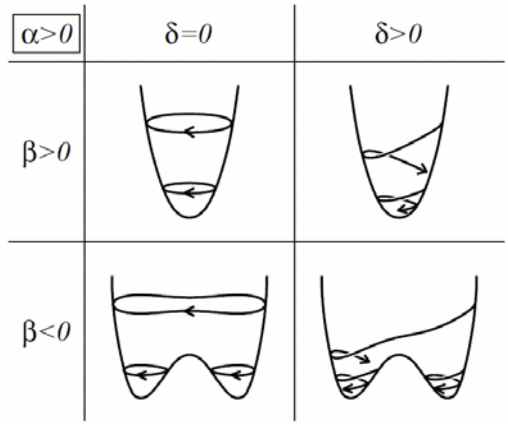

Fig 3: The shape of E(t) and schematic trajectories of the Duffing oscillator in the ($x,\dot{x},E(t)$) space for α > 0.

When δ > 0, E(t) satisfies:

$$\frac{dE(t)}{dt}=-\delta\dot{x}^2\le 0 \quad (7)$$

therefore, the trajectory of x moves on the surface of E(t) so that E(t) decreases until x converges to one of the equilibrium where $\dot{x}=0$. For α > 0, β > 0, and δ > 0, the only equilibrium is $\bar{x}\equiv(0,0)$, and E(t) satisfies

- $E(t)=0$ if and only if $x=\bar{x}$,
- $E(t)\rangle 0$ and $\dot{E}(t)\langle 0$ for $x\ne\bar{x}$.

Therefore, E(t) is a Lyapunov function and $\bar{x}$ is globally asymptotically stable in this case. On the other hand, for α > 0, β < 0, and δ > 0, there are three equilibrium as shown, two of which are at the bottoms of E(t) and one of which is at its peak. In this case, almost all the initial conditions converge to one of the equilibrium at the bottoms, except for the initial conditions on the stable manifold of the equilibrium at the peak.

The equilibrium of the Duffing oscillator for $\gamma=0$ can be obtained by substituting $\dot{x}=0$ to equation, namely,

$$x\left(\beta+\alpha x^2\right)=0 \quad (8)$$

Therefore, the point x = 0 is always an equilibrium. Moreover, when $\alpha\beta<0$ two equilibrium points $x=\pm\sqrt{-\beta/\alpha}$ appear. The stability of this equilibrium can be understood by analysing the eigenvalues of the Jacobian matrix of the equation. Equation for $\gamma=0$ can is:

$$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ -\delta\,\dot{x} - \beta x - \alpha x^3 \end{pmatrix} \quad (9)$$

and the Jacobian matrix $DF(x)$ of the right-hand side is calculated as:

$$DF(x) = \begin{pmatrix} 0 & 1 \\ -\beta - 3\alpha x^2 & -\delta \end{pmatrix} \quad (10)$$

Therefore, the eigenvalues of $DF(x)$ for the equilibrium $x = 0$ is:

$$\lambda = \frac{-\delta \pm \sqrt{\delta^2 - 4\beta}}{2} \quad (11)$$

and it is found that this equilibrium is stable for $\beta \geq 0$ and unstable for β<0. On the other hand, the eigenvalues of the equilibrium $x = \pm\sqrt{-\beta/\alpha}$ are:

$$\lambda = \frac{-\delta \pm \sqrt{\delta^2 + 8\beta}}{2} \quad (12)$$

and it is found that these equilibrium are stable for α > 0 and β < 0, and unstable for α < 0 and β > 0.

Here we consider the response of the Duffing oscillator to a weak periodic forcing. First, by applying transformations $\beta = \omega_0^2$, $\alpha \to \varepsilon\alpha$, $\gamma \to \varepsilon\gamma$ and $\delta \to \varepsilon\delta$ to equation, we obtain:

$$\ddot{x} + \omega_0^2 = \varepsilon\left(-\delta\,\dot{x} - \alpha x^3 + \gamma\cos\omega t\right) \quad (13)$$

Because $\beta = \omega_0^2 \geq 0$, describes the response of a weakly non-linear spring to a weak periodic forcing. In the following, we find an almost sinusoidal solution of frequency $\omega \cong \omega_0$. To make electrical circuits described by equation, active circuit elements with the cubic non-linear property, $i = \phi(\upsilon) = \gamma\upsilon^3 - \alpha\upsilon$, is required, where i and υ are current and voltage, respectively. In 1920s, van der Pol built the oscillator using the triode or tetrode. After Reona Esaki [1925] invented the tunnel diode in 1957, making of the van der Pol oscillator with electrical circuits has become much simpler.

Using the tunnel diode with input-output relation:

$$i = \phi_t(\upsilon) = \phi(\upsilon - E_0) + I_0 \quad (14)$$

the equation for the circuit shown is written as follows:

$$\dot{V} = \frac{1}{C}\left[-\phi(V) - W\right] \quad (15)$$

$$\ddot{W} = \frac{1}{L}V \quad (16)$$

This can be rewritten as:

$$\ddot{V} - \frac{1}{C}\left(\alpha - 3\gamma V^2\right)\dot{V} + \frac{1}{LC}V = 0 \quad (17)$$

Introducing new variables $x = \sqrt{3\gamma/\alpha V}$, $t' = t/\sqrt{LC}$ and $\varepsilon = \sqrt{L/C\alpha}$ the relation can be transformed into an equation. As shown in the previous section, when ε is large, the period of oscillation is proportional to ε. Thus, the original system has a period equal to $T\alpha \in \sqrt{LC} = L\alpha$. Because α has an order of the reciprocal of resistance r, $T\alpha\, L/r$ is obtained. L/R is the time constant of relaxation in LR circuit; therefore, the name of "relaxation oscillation" is justified. The electrical circuit elements with the non-linear property can also be realized using operational amplifiers. By this method, many researches have been done to study the non-linear dynamics in physical systems.

A response of the system to a periodic forcing with $T_{in} = 10$ and $F = 1.2$ .Van der Pol had already examined the response of the van der Pol oscillator to a periodic forcing in his paper in 1920, as follows:

$$\ddot{x} - \varepsilon\left(1 - x^2\right)\dot{x} + x = F\cos\left(\frac{2\pi t}{T_{in}}\right) \quad (18)$$

There exist two frequencies in this system, namely, the frequency of self-oscillation determined by ε and the frequency of the periodic forcing. The response of the system is shown for $T_{in} = 10$ and $F = 1.2$. It is observed that the mean period T_{out} of x often locks to mT_{in}/n, where m and n are integers. It is also known that chaos can be found in the system when the non-linearity of the system is sufficiently strong. Figure 4 shows the largest Lyapunov exponent and it is observed that chaos takes place in the narrow ε ranges.

Van der Pol and Van der Mark [1927] considered an electrical circuit composed of a resistance, a capacitance and a Ne lamp and they heard the response of the system by inserting the telephone receivers to their circuit. Besides the locking behaviours, they heard irregular noises before the period of the system jumps to the next value. They stated that this noise is a subsidiary phenomenon, but today it is thought that they heard the deterministic chaos in 1927 before Yoshisuke Ueda [1961] and Edward Lorenz [1963]. Nevertheless, Van der Pol did not identify the structure underlying a chaotic attractor in the phase space. Lorenz published a picture of a chaotic attractor in the phase space in the early 60's and Ueda did in the early 70's.

First, we introduce the Van der Pol transformation:

$$u = x\cos\omega t - \frac{\dot{x}}{\omega}\sin\omega t \quad (19)$$

$$v = -xc\sin\omega t - \frac{\dot{x}}{\omega}s\cos\omega t \quad (20)$$

where the (u, v) plane called Van der Pol plane rotates around the $\left(x, \dot{x}/\omega\right)$ plane clockwise. On this plane, sinusoidal solutions of $\left(x, \dot{x}/\omega\right)$ of frequency ω are represented as equilibrium. By differential equations and by substituting $\omega^2 - \omega_0^2 \equiv \varepsilon\Omega$ to them, we obtain:

$$\dot{u} = \frac{\varepsilon}{\omega}\left[-\Omega(u\cos\omega t - \upsilon\sin\omega t) - \omega\delta(u\sin\omega t + \upsilon\cos\omega t) + \right.$$
$$\left. + \alpha(u\cos\omega t - \upsilon\sin\omega t)^3 - \gamma\cos\omega t\right]\sin\omega t \quad (21)$$

$$\dot{\upsilon} = \frac{\varepsilon}{\omega}[-\Omega(u\cos\omega t - \upsilon\sin\omega t) - \omega\delta(u\sin\omega t + \upsilon\cos\omega t) +$$
$$+\alpha(u\cos\omega t - \upsilon\sin\omega t)^3 - \gamma\cos\omega t]\cos\omega t \qquad (22)$$

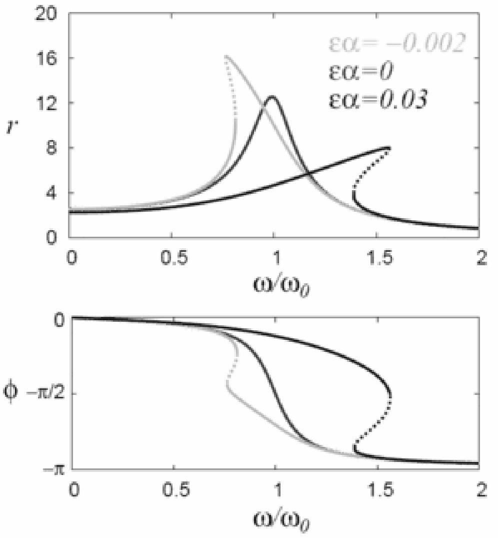

Fig. 4: The frequency response function for the Duffing oscillator for $\omega_0 = 1$, $\varepsilon\delta = 0.2$ and $\varepsilon\gamma = 2.5$. Solid and dotted lines correspond to the stable and unstable equilibrum.

Averaging equations (21) and (22) over the period $2\pi/\omega$, we obtain:

$$\dot{u} = \frac{\varepsilon}{2\omega}\left(-\omega\delta u + \Omega\upsilon - \frac{3}{4}\alpha\left(u^2+\upsilon^2\right)\upsilon\right) \qquad (23)$$

$$\dot{\upsilon} = \frac{\varepsilon}{2\omega}\left(-\Omega\upsilon - \omega\delta\upsilon + \frac{3}{4}\alpha\left(u^2+\upsilon^2\right)u - \gamma\right) \qquad (24)$$

or in polar coordinates $r = \sqrt{u^2 + \upsilon^2}$ and $\phi = arctg(\upsilon/u)$:

$$\dot{r} = \frac{\varepsilon}{2\omega}(-\omega\delta r - \gamma\sin\phi) \qquad (25)$$

$$r\dot{\phi} = \frac{\varepsilon}{2\omega}\left(-\Omega r + \frac{3}{4}\alpha r^3 - \gamma\cos\phi\right) \qquad (26)$$

By finding the equilibrium of equations (25) and (26), the response of the system to a weak periodic forcing can be analysed. When $\alpha = 0$ the frequency response function shows a peak of the usual resonance at $\omega \cong \omega_0$, and, when $\alpha \neq 0$, this peak is curved. For a hardening spring (α > 0), the peak curves to the right, and to the left for a softening spring (α < 0). By using Van der Pol plane rotating with frequency ω/k and defining $\omega^2 - k^2\omega_0^2 \equiv \varepsilon\Omega$, the k-th order subharmonics can also be analysed [Holmes and Holmes, 1981].

When the state of a dynamical system can be specified by a scalar value $x \in \Re^1$ then the system is one-dimensional. Often, only a subset of the phase line $\Re^1$ corresponds to physically meaningful states of the system, and it is often more natural to consider one-dimensional phase spaces in the form of intervals and circles. For example, the system could be a chemical reaction characterized by the concentration of a reagent or an RC-circuit characterized by the voltage across the capacitor. Notice that the former case, only non-negative values of $\Re^1$ can be used, so the phase space is [0, ∞).

One-dimensional systems are often given by the ordinary differential equation of the form $x' = f(x)$, where $x' = dx/dt$ is the derivative of the state variable x with respect to time t. This ODE is autonomous, i.e., f does not explicitly depend on the time t. The phase line of a one-dimensional ODE is partitioned by the equilibrium (points where f(x) = 0) and trajectories that connect the equilibrium. Stability of the equilibrium is determined by the directions of trajectories, which depend on the sign of the right-hand side function f(x). One does not need to solve this equation, or even know the exact details of the function f(x), to predict the dynamics of the system and its dependence on the initial condition; it is apparent from the phase portrait.

One-dimensional systems can also be given by the iterated mapping in the form:

$$x_{t+1} = f(x_t) \qquad (27)$$

where the state at time (t+1) is a function of the state at time t. Phase portraits of such systems can be quite complicated, especially when the dynamics is chaotic. One-dimensional state spaces can also be more complicated, like graphs or dendrites.

IV. CONCLUSIONS

Differential Dynamical Systems begins with coverage of linear systems, including matrix algebra; the focus then shifts to foundational material on nonlinear differential equations, making heavy use of the contraction-mapping theorem. Subsequent chapters deal specifically with dynamical systems concepts-flow, stability, invariant manifolds, the phase plane, bifurcation, chaos and Hamiltonian dynamics.

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