Applications of Fractal Signals

Dumitru ŞCHEIANU * and Ion TUTĂNESCU **

*University of Piteşti, Electronics, Communications and Computers Department,
Târgul din Vale Str., No. 1, Postal Code 110040, Piteşti - Argeş, Romania, E-Mail: dumitru.scheianu@upit.ro

* *Department of Importance,
University of Piteşti, Electronics, Communications and Computers Department,
Târgul din Vale Str., No. 1, Postal Code 110040, Piteşti - Argeş, Romania, E-Mail: tutanescu@upit.ro

***Abstract* - "Fractal" term - which in Latin language defines something fragmented anomalous - was introduced in mathematics by B. B. Mandelbrot in 1975. He avoided to define it rigorously and used it to designate some "rugged" and "self-similar" geometrical forms. Fractals were involved in the theory of chaotic dynamic systems and used to designate certain specific sets; finally, they were "captured" by geometry and remarked in tackling of the boundary problems. It proved that the fractals can be of interest even in the signal's theory.**

I. INTRODUCTION

In the category of fractals there are included also the images whose description by conventionally ways of mathematics is, in principle, impossible.
If to a 2D image (x, y) is added a third dimension (t), we have a bi-dimensional fractal signal. In such a signal, the fractal nature is manifested in (x, y) plan, having no connection with temporal dimension.
In order to consider a scalar signal as a fractal signal, the scalar signal has to fulfill three conditions related to the time domain:

- signal's chart extension on time domain has to be endless,
- scalar signal has to be continuous everywhere and
- signal has not to have a differential form on time domain.

The structure of fractal signal proposed in this paper is only one from an infinity of structures to be imagined.

II. THE CONSTRUCTION OF A FRACTAL SIGNAL

The construction of a fractal signal is realized on the base of some periodic pulses, as shown in Figure 1.
Let's consider a period of time
$T_0 = 4\tau_0$
for the first alternation (half-period) T_+ (considered centered by the axis of the time).
The first component, $x_0(t)$, is a continuous signal and its amplitude is equal with the unit. The following components, $x_1(t)$, $x_2(t)$, a.s.o. are bipolar pulses derived by division to 3 of the alternation (half-period).

The components $x_i(t)$ are described by the following mathematical equations:

$$x_0(t) = \begin{cases} 1/2^0 \;; & t \in T_+ = [-\tau_0, \tau_0] \\ 0 \;; & t \notin T_+ \end{cases} \quad (1)$$

$$x_1(t) = \begin{cases} 1/2^1 \;; & t \in T_1 = [-(1/3)\tau_0, (1/3)\tau_0] \\ -1/2^1 \;; & t \in T_+ - T_1 \\ 0 \;; & t \notin T_+ \end{cases}$$

$$x_2(t) = \begin{cases} 1/2^2 \;; & t \in T_2 = [-(7/9)\tau_0, -(5/9)\tau_0] \cup [-(1/9)\tau_0, (1/9)\tau_0] \cup [(5/9)\tau_0, (7/9)\tau_0] \\ -1/2^2 \;; & t \in T_+ - T_2 \\ 0 \;; & t \notin T_+ \end{cases}$$

$$x_i(t) = \ldots$$

(2)

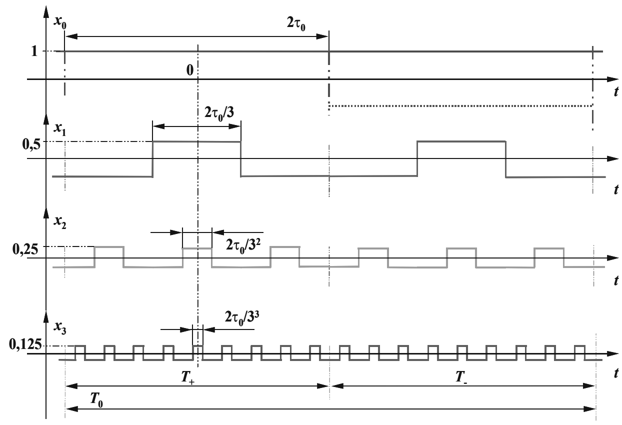

Figure 1 - Fractal signal's components.

If, for the first period, the summation of the signal's components is done according to the following relations

$$x^+(t) = \sum_{i=0}^{\infty} x_i(t) \quad ; \quad t \in T_+$$
$$x^-(t) = -\sum_{i=0}^{\infty} x_i(t - 2\tau_0) \;; \quad t \in T_- \quad (3)$$

and the result is divided into periods, we obtain the fractal signal:

$$s_f(t) = \left[x^+(t) + x^-(t)\right]_{T_0} \quad (4)$$

The evolution of signal's aspect during its making-up is shown in Figure 2.
The superscripts "+" and "-" assigned to function x(t),

$x^+(t)$, respective $x^-(t)$, say that these are defined on the half-periods T_+ and T_-, presented in the Figure 1.

The described fractal signal does not belong to the class of the functions defined in this paper as signals. Its continuity and non-differentiability are thoroughly remarked from its making-up process; also, it is remarked that it integrates a infinity of first rang discontinuity points.

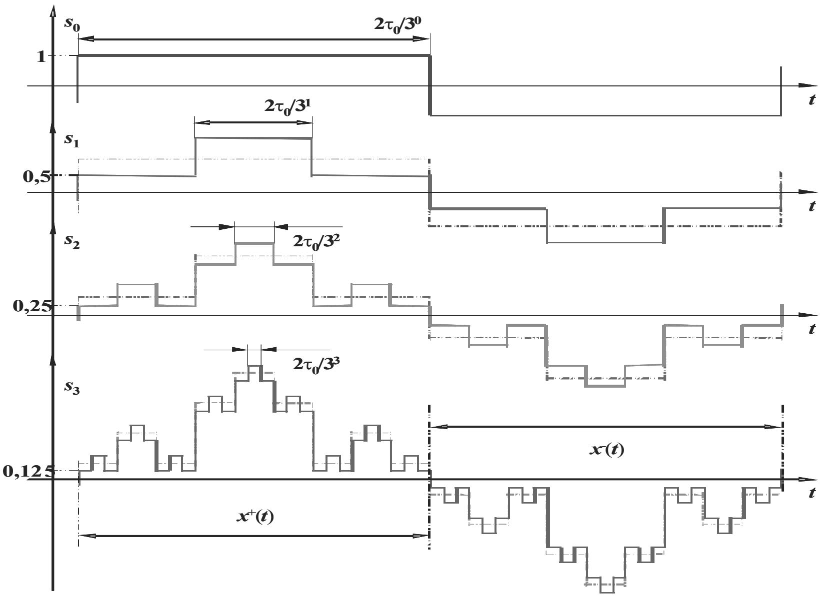

Figure 2 - Fractal signal's making-up process.

Every time this signal's amplitude belongs to the interval [-2, 2]. Clearly, its contour length comprises two constructional parts:

- one of them, on the time axis, being finite,
- the other one, made of the amplitudes (the sum of amplitude "jumps"), being infinite.

Over a period T_0 this length is:

$$L = (T_0 + 4u) + 2\sum_{n=0}^{\infty}\left(\frac{3}{2}\right)^n u = L_\infty \qquad (5)$$

where u is the unit amplitude.

Note: The nature of fractal signal is given by the $x^+(t)$ function. Considering function x(t) divided into periods, we can conclude that some fractal signals could be "everywhere" and others exist only on given time intervals.

III. SPECTRAL ANALYSIS OF FRACTAL SIGNAL

From spectral point of view, being conceived as a periodical function, to this type of signal corresponds a spectrum made from lines. First of all, for its determination, we will proceed to the dividing into periods of $x_i^+(t)$:

$$[x_0(t)]_{T_0}^+ = \begin{cases} \sum_{n=-\infty}^{\infty} x_0(t - nT_0); & t \in [T_+]_{T_0} \\ 0 & ; \quad t \notin [T_+]_{T_0} \end{cases}$$

$$[x_1(t)]_{T_0}^+ = \begin{cases} \sum_{n=-\infty}^{\infty} x_1(t - nT_0); & t \in [T_+]_{T_0} \\ 0 & ; \quad t \notin [T_+]_{T_0} \end{cases} \qquad (6)$$

$$[x_i(t)]_{T_0}^+ = \ldots$$

The next step consists in determination of Amplitude Spectral Density Function, ASDF:

$$X_0^+(\omega) = F\{[x_0(t)]_{T_0}^+\} = \pi Sa(\omega\tau_0)\sum_{n=-\infty}^{\infty}\delta\left(\omega - n\frac{2\pi}{T_0}\right) \qquad (7)$$

$$X_1^+(\omega) = F\{[x_1(t)]_{T_0}^+\} = \frac{2\pi}{4\tau_0}\left[\frac{4\tau_0}{(2*3)^1}Sa\left(\omega\frac{\tau_0}{3}\right) - \frac{\tau_0}{2^0}Sa(\omega\tau_0)\right]\sum_{n=-\infty}^{\infty}\delta\left(\omega - n\frac{2\pi}{T_0}\right)$$

$$X_2^+(\omega) = F\{[x_2(t)]_{T_0}^+\} = \frac{2\pi}{4\tau_0}\left[\frac{4\tau_0}{(2*3)^2}\left(\cos\omega\frac{-2\tau_0}{3} + 1 + \cos\omega\frac{2\tau_0}{3}\right)Sa\left(\omega\frac{\tau_0}{3^2}\right) - \frac{\tau_0}{2^1}Sa(\omega\tau_0)\right]\times$$
$$\times\sum_{n=-\infty}^{\infty}\delta\left(\omega - n\frac{2\pi}{T_0}\right)$$

$$X_3^+(\omega) = F\{[x_3(t)]_{T_0}^+\} = \frac{2\pi}{4\tau_0}\left[\frac{4\tau_0}{(2*3)^3}\left(\cos\omega\frac{-8\tau_0}{3^2} + \cos\omega\frac{-6\tau_0}{3^2} + \cos\omega\frac{-4\tau_0}{3^2} + \cos\omega\frac{-2\tau_0}{3^2} + \right.\right.$$
$$\left.\left. + 1 + \cos\omega\frac{2\tau_0}{3^2} + \cos\omega\frac{4\tau_0}{3^2} + \cos\omega\frac{6\tau_0}{3^2} + \cos\omega\frac{8\tau_0}{3^2}\right)Sa\left(\omega\frac{\tau_0}{3^3}\right) - \frac{\tau_0}{2^2}Sa(\omega\tau_0)\right]\sum_{n=-\infty}^{\infty}\delta\left(\omega - n\frac{2\pi}{T_0}\right)$$

$$X_i^+(\omega) = \frac{2\pi}{4\tau_0}\left[\left(\frac{4\tau_0}{(2*3)^i}\sum_{m=-l}^{l}\cos m\omega\frac{2\tau_0}{3^i}\right)Sa\left(\omega\frac{\tau_0}{3^i}\right) - \frac{\tau_0}{2^{i-1}}Sa(\omega\tau_0)\right]\sum_{n=-\infty}^{\infty}\delta\left(\omega - n\frac{2\pi}{T_0}\right); \quad l = \frac{3^{i-1}-1}{2}$$

(8)

The determination of Fourier Transform FT for $[x_i(t)]_{T_0}^-$ functions it will be done similarly. Having in view that these are the result of inversion and translation of

$\left[x_i(t)\right]^{+}_{T_0}$ functions, we have:

$$X_i^{-}(\omega) = -X_i^{+}(\omega)e^{-j\omega\frac{T_0}{2}} \qquad (9)$$

By summation of components' contributions on both semi-periods, the spectrum generated by a component $\left[x_i(t)\right]_{T_0}$ is:

$$X_i(\omega) = F\left\{\left[x_i(t)\right]^{+}_{T_0} + \left[x_i(t)\right]^{-}_{T_0}\right\} = \\ \vdots = X_i^{+}(\omega)\left[1 - e^{-j\omega\frac{T_0}{2}}\right] \qquad (10)$$

The spectrum of $s_f(t)$ is the summation of components' spectra:

$$S_f(\omega) = \sum_{i=0}^{\infty} X_i^{+}(\omega)\left[1 - e^{-j\omega\frac{T_0}{2}}\right] \qquad (11)$$

The terms containing $Sa(\omega\tau_0)$ function (see relations 8) are part of the following expression, $A(\omega)$:

$$A(\omega) = \left(\pi - \sum_{i=1}^{\infty}\frac{\pi}{2^i}\right)Sa(\omega\tau_0)\times \\ \times \sum_{n=-\infty}^{\infty}\delta\left(\omega - n\frac{2\pi}{T_0}\right)\left(1 - e^{-j\omega\frac{T_0}{2}}\right) = 0 \qquad (12)$$

This expression is equal to zero due to the first factor. Thus, the relation (11) should be written as we can see below:

$$S_f(\omega) = \sum_{i=1}^{\infty}\left[\left(\frac{2\pi}{(2*3)^i}\sum_{m=-1}^{1}\cos m\omega\frac{2\tau_0}{3^i}\right)Sa\left(\omega\frac{\tau_0}{3^i}\right)\right]\sum_{n=-\infty}^{\infty}\delta\left(\omega - n\frac{2\pi}{T_0}\right)\left(1 - e^{-j\omega\frac{T_0}{2}}\right)$$

(13)

Taking into consideration that this relation is unequal to zero on frequencies

$$\omega_n = n\frac{2\pi}{T_0} \qquad (14)$$

and that

$$1 - e^{-j\omega_n\frac{T_0}{2}} = 1 - e^{-jn\pi} = \ldots 0,\ 2,\ 0,\ \ldots \qquad (15)$$

the signal's spectral function may be written as the multiplication of convolute function $S_{fc}(\omega)$ by Dirac impulse series

$$S_f(\omega) = S_{fc}(\omega)\sum_{n=-\infty}^{\infty}\delta\left[\omega - (2n+1)\frac{\pi}{2\tau_0}\right] \qquad (16)$$

The expression of convolute function is:

$$S_{fc}(\omega) = 2\sum_{i=1}^{\infty}\left[\left(\frac{2\pi}{(2*3)^i}\sum_{m=-1}^{1}\cos m\omega\frac{2\tau_0}{3^i}\right)\times \\ \times Sa\left(\omega\frac{\tau_0}{3^i}\right)\right] \qquad (17)$$

Because

$$\sum_{m=-1}^{1}\cos m\omega\frac{2\tau_0}{3^i} = \frac{\sin\omega\frac{\tau_0}{3}}{\sin\omega\frac{\tau_0}{3^i}} \qquad (18)$$

we can write:

$$S_{fc}(\omega) = 2\sum_{i=1}^{\infty}\left(\frac{2\pi}{(2*3)^i}\frac{\sin\omega\frac{\tau_0}{3}}{\sin\omega\frac{\tau_0}{3^i}}\frac{\sin\omega\frac{\tau_0}{3^i}}{\omega\frac{\tau_0}{3^i}}\right) = \\ = \frac{4\pi}{3}Sa\left(\omega\frac{\tau_0}{3}\right) \qquad (19)$$

Finally, ASDF of the fractal signal already described is done by the relation

$$S_f(\omega) = \frac{4\pi}{3}Sa\left(\omega\frac{\tau_0}{3}\right)\sum_{n=-\infty}^{\infty}\delta\left[\omega - (2n+1)\frac{2\pi}{4\tau_0}\right] \qquad (20)$$

and is presented in Figure 3.

The fractal signal being even, its Fourier Transform is real. It can be noticed that to the rugged form from the temporal domain corresponds a conventional convolute function.

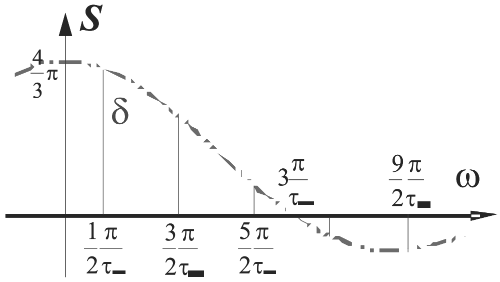

Figure 3 - Spectral presentation of the fractal signal.

IV. PROBABILITY ANALYSIS OF FRACTAL SIGNAL

It is interesting to analyze this type of signal from probabilistic point of view. Therefore, if we consider the set of time intervals $\tau_0/3^i$ whereon s_i signal doesn't change its value (see Figure 2) and accept the idea that their distribution is uniform on t axis, it is possible to define fractal signal's probability density function p(s) as a weight of the temporal intervals length in which the signal takes a certain value on the temporal axis' length.

By successive summation, our result is the percentile P(s), as we can see in Figure 4. Because the structure of fractal signal is similar in both positive and negative domain, these function's form were presented in Fig. 4 in positive domain only.

Given the signal's nature, the probability density function is formed by weighted Dirac impulses, and the percentile by a multi-stage function.

Construction of the probability density function begins from the case of s_1 density probability function (which presents two lines of amplitude 0,5δ) and is realized according to the following rule: every line corresponding to the signal s_i generates two lines for the signal s_{i+1}, delayed with $\pm 2^{-I}$ from the line corresponding to the signal s_i . Their size is in a ratio of ½, the biggest line being that one which is situated on s axis.

When the index i is in infinite sub-domain, the lines' amplitudes which forms the probability density function becomes finite, and the distance between them becomes infinitesimal.

The percentile function, beyond the aspect relatively common, does not show a characteristic of the fractal signal, because its bent can vary from a line to another; when the index i comes into infinite values sub-domain, P(s) becomes a non-differentiable function.

With other words, to the fractal signal we can associate - by construction - the pair formed by the probability density function and the percentile function; but, going from one to other function is not possible.

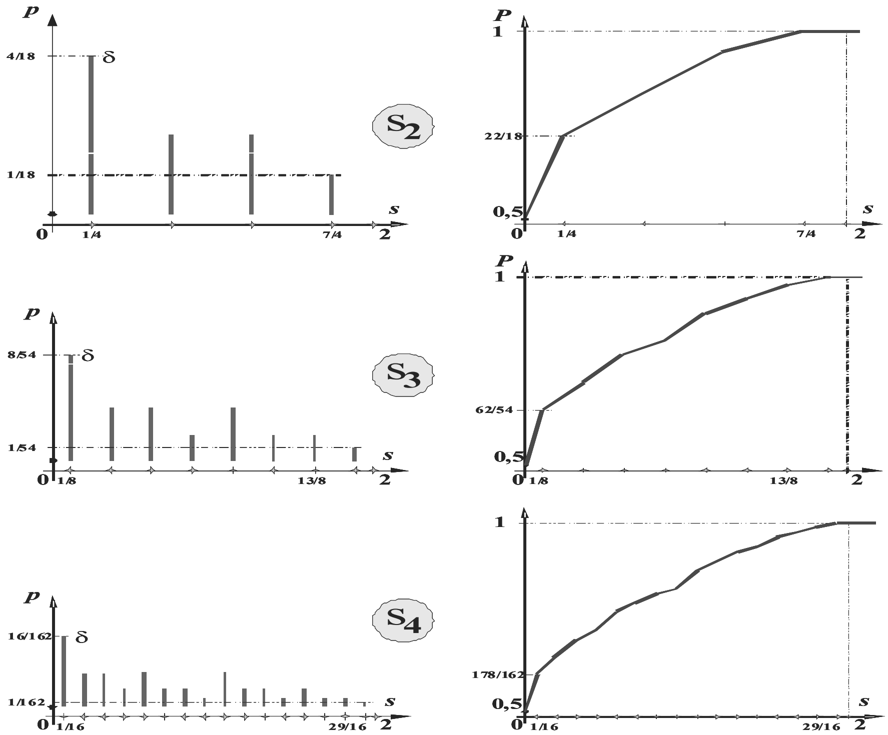

Figure 4 - The construction of probability density function and percentile function associated to the fractal signal.

To determine the temporal moments is relative easily. So, starting from the construction of the signal, we can directly affirm that the average value is zero and, consequently, the centered moments will be equal with the direct moments. Therefore, the dispersion will be equal to the second order moment and - because of the signal's similitude on the space of T_+ and T_+ - it may be calculated on the positive alternation

$$\sigma^2 = \frac{1}{2\tau_0}\int_0^{2\tau_0} s_f^2(t)dt = \frac{1}{2\tau_0}\sum_{i=0}^{\infty}\left(\frac{1}{2^i}\right)^2 \qquad (21)$$

and it has the finite value $2/(3\tau_0)$.

V. CONCLUSIONS

A thought-provoking problem is to frame this signal into an ordinary class.

If the index i is finite, the signal is determined, this meaning that the signal is altogether knowable, on its entire evolution.

If the index i is situated into infinite values sub-domain, the signal is one of fractal type and the associated probability density function and percentile function take some specific characteristics from the fractal signal.

This signal isn't a stochastic process, its values domain being not a set of random variables. But, because it is possible to be known in its any point, it may be assimilated to a stochastic process' trajectory. Anyway, there is a very important difference.

It is well known that a certain trajectory of a stochastic process can't be recovered on the base of an value set, as much and as well this set is chosen. In the case of the fractal signals, besides a values set, it is known a very important thing – the construction's algorithm. The problem is to find the best way for enlarge its cognition in comparison with the case of a certain trajectory of a stochastic process.

Some applications of the fractal signals:

The study of fractal signals has proved to be important in several scientific domains. The examples of processes described by such functions include: bio-signals (by example, heartbeat oscillations), elementary cellular automata, speech signals, communications (by example, network traffic), etc. These possible applications are shortly presented below:

- Using analysis methods for quantify long-range power-law correlations in noisy heartbeat fluctuations, the human heartbeat dynamics under healthy and pathologic conditions can be analyzed. There were found power-law correlations which indicate the presence of scale-invariant, fractal structures in the human heartbeat. There is recent work that quantifies multi-fractal features in cascades. The multi-fractal structure of healthy dynamics is lost with congestive heart failure [5]. These analytic tools may be used also on a wide range of other physiologic signals and the findings may lead to new diagnostic applications.

- Computing the power spectra of the one-dimensional elementary cellular automata, some interesting results can be obtained: on the one hand, the analysis reveals that one automaton displays 1/f spectra though considered as trivial; on the other hand, various automata classified as chaotic or complex display no 1/f spectra. Generalizing the Sierpinski signal to a class of fractal signals that are tailored to produce 1/f spectra, a model of the one-dimensional elementary cellular automata results can be obtained. From the widespread occurrence of elementary cellular automata patterns in chemistry, physics, and computer sciences, there are various candidates to show spectra similar to these results [6].

- Fractals can model many classes of time series data. An important characteristic is fractal dimension that represents the complexity of the time series data. In particular, in analysis of speech signal, the fractal dimension represents a powerful tool for identification of some key features of speech signal (vocals, consonants, transition from vocal to consonant and vice versa). For measuring the fractal dimension of speech signals several algorithms can be used (Higuchi algorithm, morphological covering, wavelet based method, etc.). The multi-scale fractal dimension can potentially be used to discriminate among phonetic classes, with applications in automatic speech recognition [7].

- Fractal models have made a major impact in the area of communications, particularly in the area of computer data networks. Several studies have demonstrated that network traffic loads exhibit fractal properties. These properties strongly influence network performance. For instance, performance predictions based on classical traffic models are often far too optimistic when compared against actual performance with real data. Fractal traffic models allow exciting new insights into network behavior and promise new algorithms for network data prediction and control [8].

REFERENCES

[1] D. Şcheianu, " Semnale, componente, circuite şi sisteme", Partea I, Editura ALDO, Bucureşti, ISBN 973-98737-3-1, 2005;

[2] D. Şcheianu, "Compendiu de teoria semnalelor în locație". Editura ALDO, Bucureşti, ISBN 973-98737-0-7, 1998;

[3] A. Georgescu, "Sinergetica. Solitoni. Fractali. Haos determinist. Turbulența." Universitatea din Timişoara, Monografii matematice nr. 43, 1992;

[4] I. Tutănescu, "Generation of chaotic signals using continuous time dynamic systems and its applications", University of Pitesti Scientific Journal - Electronics and Computer Science, 2001;

[5] P. C. Ivanov a.o., "From 1/f noise to multifractal cascades in heartbeat dynamics, Chaos", Sept. 2001;

[6] J. Nagler, J. C. Claussen, "1/f spectra in elementary cellular automata and fractal signals", Physical Review E 71, 067103, 2005;

[7] P. Maragos, A. Potamianos, "Fractal dimensions of speech sounds: Computation and application to automatic speech recognition", Journal of Acoustical Society of America, No. 105 (3), March 1999;

[8] R.H. Riedi, M.S. Crouse, V.J. Ribeiro, R.G. Baraniuk, "A Multifractal Wavelet Model with Application to Network Traffic", IEEE Transactions on Information Theory, Vol 45, No. 3 April 1999.